

Extensions of Banach Lie-Poisson spaces

Anatol Odziejewicz¹ and Tudor S. Ratiu²

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Abstract

The extension of Banach Lie-Poisson spaces is studied and linked to the extension of a special class of Banach Lie algebras. The case of W^* -algebras is given particular attention. Semidirect products and the extension of the restricted Banach Lie-Poisson space by the Banach Lie-Poisson space of compact operators are given as examples.

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1 Introduction

The dual of any finite dimensional Lie algebra carries a linear Poisson bracket, called *Lie-Poisson structure*, which is pervasive in classical mechanics. Many Hamiltonian systems, such as the free or heavy rigid body equations, the finite and periodic Toda lattice, the geodesics on quadrics, or the Neumann and Rosochatius system, have alternate non-canonical descriptions in Lie-Poisson formulation. Formally, several evolutionary

¹Institute of Physics, University of Bialystok, Lipowa 41, PL-15424 Bialystok, Poland. aodziejew@labfiz.uwb.edu.pl

²Centre Bernoulli, École Polytechnique Fédérale de Lausanne. CH-1015 Lausanne. Switzerland. Tudor.Ratiu@epfl.ch

partial differential equations, such as the ideal non viscous fluid, ideal magnetohydrodynamics, the Poisson-Vlasov, Korteweg-de Vries, Kadomtsev-Petviashvili, or the linear and nonlinear wave and Schrödinger equations also have Lie-Poisson formulations (see, for example, Marsden and Ratiu [1994], Dubrovin, Krichever, and Novikov [2001] and references therein). While these applications of this linear Poisson structure emerged in the work of the past decades, the structure itself goes back to Lie [1890], who introduced it simultaneously with the concept of Lie algebra. The theory of Lie-Poisson spaces in finite dimensions is complete and is part of Poisson geometry (see, for example, Weinstein [1983], [1998], Marsden and Ratiu [1994], Vaisman [1994]). In infinite dimensions a general theory is lacking. Chernoff and Marsden [1974] represents the first systematic attempt to lay the foundations of Banach symplectic geometry.

Motivated by our understanding of infinite dimensional Hamiltonian systems, questions surrounding the notion of momentum map, and problems in quantum mechanics including the theory of coherent states, Odziejewicz and Ratiu [2003] propose a definition of Banach Lie-Poisson space and link it to classical and quantum reduction, the theory of W^* -algebras, and momentum maps in infinite dimensions. Banach Lie-Poisson spaces naturally appear in this context as preduals of Banach Lie algebras. For example, the spaces of compact and trace class operators on a complex separable Hilbert space carry a natural Lie-Poisson bracket. The present work develops further this point of view by addressing the fundamental question of construction of new Banach Lie-Poisson spaces out of given ones. One such scheme is given by the method of extensions.

The problem of extension in various categories plays a central role in the understanding of its objects and morphisms. It gives a method to construct new objects out of old ones whose properties are then well understood. The category of Banach Lie-Poisson spaces is no exception. The goal of this paper is to present the theory of extensions for Banach Lie-Poisson spaces and to give several mathematically and physically relevant examples.

The paper is organized as follows. In §2 the necessary information on Banach Lie-Poisson spaces found in Odziejewicz and Ratiu [2003] is collected. Only some definitions and theorems necessary for the subsequent development are given. With this background, the theory of exact sequences of Banach Lie-Poisson spaces is presented in §3. It is shown that exactness in this category is equivalent to exactness of the dual Banach Lie algebra sequence in the subcategory of Banach Lie algebras admitting a predual. Special attention is devoted to the important case of the Banach Lie-Poisson spaces that are preduals of W^* -algebras. It is shown that if the dual sequence is exact in the category of W^* -algebras, then the exact sequence is necessarily that of a direct sum of Banach Lie-Poisson spaces. Extensions of Banach Lie algebras are discussed in §4. All possible brackets on a Banach space direct sum of Banach Lie algebras are characterized. With this preparation, §5 presents all extensions of Banach Lie-Poisson spaces underlying a Banach space direct sum. Semidirect products of Banach Lie-Poisson spaces with cocycles are a particular case of this theory. Even more special, the case of the predual of the semidirect product of a W^* -algebra with a representation is treated in detail. The example of the extension of the restricted Banach Lie-Poisson space by the space of compact operators, important in the theory of loop groups (Pressley and Segal [1986], Wurzbacher [2001]), is also worked out.

2 Banach Lie algebras and Lie-Poisson spaces

This section briefly reviews the minimal background from Odziejewicz and Ratiu [2003] necessary for the rest of the paper. No proofs will be given here since they can be found in the aforementioned paper.

Given a Banach space \mathfrak{b} , the notation \mathfrak{b}^* will always be used for the Banach space dual to \mathfrak{b} . For $x \in \mathfrak{b}^*$ and $b \in \mathfrak{b}$, the notation $\langle x, b \rangle$ means the value of x on b . Thus $\langle \cdot, \cdot \rangle : \mathfrak{b}^* \times \mathfrak{b} \rightarrow \mathbb{R}$ will denote the natural bilinear continuous duality pairing between \mathfrak{b} and its dual \mathfrak{b}^* . The notation \mathfrak{b}_* will be reserved for the *predual* of \mathfrak{b} , that is, \mathfrak{b}_* is a Banach space whose dual is \mathfrak{b} . The predual is not unique, in general. Note also that $\mathfrak{b}_* \hookrightarrow \mathfrak{b}^*$ canonically and that \mathfrak{b}_* is a closed subspace of \mathfrak{b}^* .

Recall that a **Banach Lie algebra** $(\mathfrak{g}, [\cdot, \cdot])$ is a Banach space that is also a Lie algebra such that the Lie bracket is a bilinear continuous map $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. Thus the adjoint and coadjoint maps $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}$, $\text{ad}_x y := [x, y]$, and $\text{ad}_x^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ are also continuous for each $x \in \mathfrak{g}$.

A **Banach Poisson manifold** is a pair $(P, \{\cdot, \cdot\})$ consisting of a smooth (real or complex) Banach manifold P and a bilinear operation $\{\cdot, \cdot\}$ on the ring $C^\infty(P)$ such that:

- $(C^\infty(P), \{\cdot, \cdot\})$ is Lie algebra,
- the Leibniz identity holds: $\{fg, h\} = f\{g, h\} + \{f, h\}g$ for all $f, g, h \in C^\infty(P)$,
- for each $f \in C^\infty(P)$, the derivation $X_f := \{\cdot, f\}$ which is, in general, a section of $T^{**}P$, is a vector field on P .

A **Banach Lie-Poisson space** $(\mathfrak{b}, \{\cdot, \cdot\})$ is defined to be a (real or complex) Banach space \mathfrak{b} that is also a Poisson manifold satisfying the additional condition that its dual $\mathfrak{b}^* \subset C^\infty(\mathfrak{b})$ is a Banach Lie algebra under the Poisson bracket operation. The following characterization is crucial throughout this paper.

Theorem 2.1 *The Banach space \mathfrak{b} is a Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot, \cdot\})$ if and only if its dual \mathfrak{b}^* is a Banach Lie algebra $(\mathfrak{b}^*, [\cdot, \cdot])$ satisfying $\text{ad}_x^* \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{b}^{**}$ for all $x \in \mathfrak{b}^*$. Moreover, the Poisson bracket of $f, g \in C^\infty(\mathfrak{b})$ is given by*

$$\{f, g\}(b) = \langle [Df(b), Dg(b)], b \rangle, \quad (2.1)$$

where $b \in \mathfrak{b}$ and D denotes the Fréchet derivative. If $h \in C^\infty(\mathfrak{b})$, the associated Hamiltonian vector field is given by

$$X_h(b) = -\text{ad}_{Dh(b)}^* b. \quad (2.2)$$

A **morphism** between two Banach Lie-Poisson spaces \mathfrak{b}_1 and \mathfrak{b}_2 is a continuous linear map $\phi : \mathfrak{b}_1 \rightarrow \mathfrak{b}_2$ that preserves the Poisson bracket, that is,

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi$$

for any $f, g \in C^\infty(\mathfrak{b}_2)$. Such a map ϕ is also called a **linear Poisson map**. Define the **category \mathfrak{B} of Banach Lie-Poisson spaces** as the category whose objects are the Banach Lie-Poisson spaces and whose morphisms are the linear Poisson maps.

Let \mathfrak{L} denote the **category of Banach Lie algebras**; its objects are Banach Lie algebras and its morphisms are continuous Lie algebra homomorphisms.

Denote by \mathfrak{L}_0 the following subcategory of \mathfrak{L} . An object of \mathfrak{L}_0 is a Banach Lie algebra \mathfrak{g} admitting a predual \mathfrak{g}_* , that is, $(\mathfrak{g}_*)^* = \mathfrak{g}$, and satisfying $\text{ad}_{\mathfrak{g}}^* \mathfrak{g}_* \subset \mathfrak{g}_*$ where ad^* is the coadjoint representation of \mathfrak{g} on \mathfrak{g}^* ; recall that \mathfrak{g}_* is a closed subspace of \mathfrak{g}^* . A morphism in the category \mathfrak{L}_0 is a Banach Lie algebra homomorphism $\psi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that the dual map $\psi^* : \mathfrak{g}_2^* \rightarrow \mathfrak{g}_1^*$ preserves at least one choice of the corresponding preduals, that is, $\psi^* : (\mathfrak{g}_2)_* \rightarrow (\mathfrak{g}_1)_*$, where $(\mathfrak{g}_i)_*$ is one possible predual of \mathfrak{g}_i for $i = 1, 2$. Let \mathfrak{L}_{0u} be the subcategory of \mathfrak{L}_0 whose objects have a *unique* predual.

Theorem 2.2 *There is a contravariant functor $\mathfrak{F} : \mathfrak{B} \rightarrow \mathfrak{L}_0$ defined by $\mathfrak{F}(\mathfrak{b}) = \mathfrak{b}^*$ and $\mathfrak{F}(\phi) = \phi^*$. On the subcategory $\mathfrak{F}^{-1}(\mathfrak{L}_{0u}) \subset \mathfrak{B}$ this functor is invertible. The inverse of \mathfrak{F} is given by $\mathfrak{F}^{-1}(\mathfrak{g}) = \mathfrak{g}_*$ and $\mathfrak{F}^{-1}(\psi) = \psi^*|_{(\mathfrak{g}_2)_*}$, where $\psi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$.*

The internal structure of morphisms in \mathfrak{B} is given by the following results.

Proposition 2.3 *Let $\phi : \mathfrak{b}_1 \rightarrow \mathfrak{b}_2$ be a linear Poisson map between Banach Lie-Poisson spaces and assume that $\text{im } \phi$ is closed in \mathfrak{b}_2 . Then the Banach space $\mathfrak{b}_1 / \ker \phi$ is predual to $\mathfrak{b}_2^* / \ker \phi^*$, that is, $(\mathfrak{b}_1 / \ker \phi)^* \cong \mathfrak{b}_2^* / \ker \phi^*$. In addition, $\mathfrak{b}_2^* / \ker \phi^*$ is a Banach Lie algebra satisfying the condition $\text{ad}_{[x]}^* (\mathfrak{b}_1 / \ker \phi) \subset \mathfrak{b}_1 / \ker \phi$ for all $[x] \in \mathfrak{b}_2^* / \ker \phi^*$ and $\mathfrak{b}_1 / \ker \phi$ is a Banach Lie-Poisson space. Moreover, the following properties hold:*

- (i) *the quotient map $\pi : \mathfrak{b}_1 \rightarrow \mathfrak{b}_1 / \ker \phi$ is a surjective linear Poisson map;*
- (ii) *the map $\iota : \mathfrak{b}_1 / \ker \phi \rightarrow \mathfrak{b}_2$ defined by $\iota([b]) := \phi(b)$, where $b \in \mathfrak{b}_1$ and $[b] \in \mathfrak{b}_1 / \ker \phi$ is an injective linear Poisson map;*
- (iii) *the decomposition $\phi = \iota \circ \pi$ into a surjective and an injective linear Poisson map is valid.*

Proposition 2.3 reduces the study of linear Poisson maps with closed range between Banach Lie-Poisson spaces to the study of surjective and injective linear Poisson maps, which is carried out in the next propositions.

Proposition 2.4 *Let $(\mathfrak{b}_1, \{\cdot, \cdot\})$ be a Banach Lie-Poisson space and let $\pi : \mathfrak{b}_1 \rightarrow \mathfrak{b}_2$ be a continuous linear surjective map onto the Banach space \mathfrak{b}_2 . Then \mathfrak{b}_2 carries a Banach Lie-Poisson structure such that π is a linear Poisson map if and only if $\text{im } \pi^* \subset \mathfrak{b}_1^*$ is closed under the Lie bracket $[\cdot, \cdot]_1$ of \mathfrak{b}_1^* . This Banach Lie-Poisson structure on \mathfrak{b}_2 is unique and it is called the **coinduced** structure by the mapping π . The map $\pi^* : \mathfrak{b}_2^* \rightarrow \mathfrak{b}_1^*$ is a Banach Lie algebra morphism whose dual $\pi^{**} : \mathfrak{b}_1^{**} \rightarrow \mathfrak{b}_2^{**}$ maps \mathfrak{b}_1 into \mathfrak{b}_2 .*

Proposition 2.5 *Let \mathfrak{b}_1 be a Banach space, $(\mathfrak{b}_2, \{\cdot, \cdot\}_2)$ be a Banach Lie-Poisson space, and $\iota : \mathfrak{b}_1 \rightarrow \mathfrak{b}_2$ be an injective continuous linear map with closed range. Then \mathfrak{b}_1 carries a unique Banach Lie-Poisson structure such that ι is a linear Poisson map if and only if $\ker \iota^*$ is an ideal in the Banach Lie algebra \mathfrak{b}_2^* . This Banach Lie-Poisson structure on \mathfrak{b}_1 is unique and it is called the structure **induced** by the mapping ι . The map $\iota^* : \mathfrak{b}_2^* \rightarrow \mathfrak{b}_1^*$ is a Banach Lie algebra morphism whose dual $\iota^{**} : \mathfrak{b}_1^{**} \rightarrow \mathfrak{b}_2^{**}$ maps \mathfrak{b}_1 into \mathfrak{b}_2 .*

For later applications we shall need also the notion of the product of Banach Poisson manifolds.

Theorem 2.6 *Given the Banach Poisson manifolds $(P_1, \{, \}_1)$ and $(P_2, \{, \}_2)$ there is a unique Banach Poisson structure $\{, \}_{12}$ on the product manifold $P_1 \times P_2$ such that:*

- (i) *the canonical projections $\pi_1 : P_1 \times P_2 \rightarrow P_1$ and $\pi_2 : P_1 \times P_2 \rightarrow P_2$ are Poisson maps;*
- (ii) *$\pi_1^*(C^\infty(P_1))$ and $\pi_2^*(C^\infty(P_2))$ are Poisson commuting subalgebras of $C^\infty(P_1 \times P_2)$.*

*This unique Poisson structure on $P_1 \times P_2$ is called the **product** Poisson structure and its bracket is given by the formula*

$$\{f, g\}_{12}(p_1, p_2) = \{f_{p_2}, g_{p_2}\}_1(p_1) + \{f_{p_1}, g_{p_1}\}_2(p_2), \quad (2.3)$$

where $f_{p_1}, g_{p_1} \in C^\infty(P_2)$ and $f_{p_2}, g_{p_2} \in C^\infty(P_1)$ are the partial functions given by $f_{p_1}(p_2) = f_{p_2}(p_1) = f(p_1, p_2)$ and similarly for g .

3 Exact sequences of Banach Lie-Poisson spaces

In this section we will study exact sequences in the categories presented in the previous section. Exactness in the categories \mathfrak{L} , \mathfrak{L}_0 , \mathfrak{L}_{0u} , and \mathfrak{B} is defined in the following way.

Definition 3.1 *A sequence of Banach Lie algebras*

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0 \quad (3.1)$$

*is **exact** if it is exact as a sequence in the category of Banach spaces and all maps are Banach Lie algebra homomorphisms. The Lie algebra \mathfrak{g} is said to be an **extension** of \mathfrak{h} by \mathfrak{n} .*

Definition 3.2 *In the categories \mathfrak{L}_0 (respectively \mathfrak{L}_{0u}) the sequence (3.1) is **exact** if it is exact in the category \mathfrak{L} and the duals of the maps in the sequence preserve at least one choice of (respectively the uniquely associated) predual spaces, that is $\iota(\mathfrak{g}_*) \subset \mathfrak{n}_*$ and $\pi_*(\mathfrak{h}_*) \subset \mathfrak{g}_*$, where \mathfrak{n}_* , \mathfrak{g}_* , \mathfrak{h}_* are preduals of \mathfrak{n} , \mathfrak{g} , \mathfrak{h} respectively. Like in the previous case, \mathfrak{g} is said to be an **extension** of \mathfrak{h} by \mathfrak{n} in these two categories.*

Definition 3.3 *A sequence of Banach Lie-Poisson spaces*

$$0 \longrightarrow \mathfrak{a} \xrightarrow{j} \mathfrak{b} \xrightarrow{p} \mathfrak{c} \longrightarrow 0 \quad (3.2)$$

*is **exact** if it is exact as a sequence in the category of Banach spaces and all maps are linear Poisson maps. The Banach Lie-Poisson space \mathfrak{b} is said to be an **extension** of \mathfrak{c} by \mathfrak{a} .*

The goal of this section is to study under what conditions the functor \mathfrak{F} preserves exactness. The answer is given by the following theorem.

Theorem 3.4 *The Banach spaces \mathfrak{a} , \mathfrak{b} , \mathfrak{c} form an exact sequence (3.2) of Banach Lie-Poisson spaces if and only if their duals $\mathfrak{n} := \mathfrak{c}^*$, $\mathfrak{g} := \mathfrak{b}^*$, $\mathfrak{h} := \mathfrak{a}^*$ form an exact sequence of Banach Lie algebras (3.1) in the category \mathfrak{L}_0 , where $\iota := p^*$ and $\pi := j^*$. In particular, if \mathfrak{g} is the direct sum $\mathfrak{g} \cong \mathfrak{n} \oplus \mathfrak{h}$ of Banach Lie algebras with ι the inclusion of the first component and π the projection on the second component, then \mathfrak{b} can be chosen as the direct sum $\mathfrak{a} \oplus \mathfrak{c}$ of the Banach Lie-Poisson spaces \mathfrak{a} and \mathfrak{c} with j the inclusion on the first component and p the projection on the second component.*

For the proof we shall need a few preparatory lemmas. The subjects of two first lemmas are well known facts but we include them for the sake of completeness.

Lemma 3.5 *Let U and W be Banach spaces. Then one has the canonical isomorphism*

$$(U \oplus W)^* \cong U^* \oplus W^*.$$

Proof. To $f \in (U \oplus W)^*$ associate the pair $(f|_U, f|_W) \in U^* \oplus W^*$. This map is clearly linear and continuous if on the direct sum one takes the norm given by the sum of norms in each component. The map that associates to $(\gamma, \alpha) \in U^* \oplus W^*$ the functional $\gamma + \alpha \in (U \oplus W)^*$, defined by $(\gamma + \alpha)(c, a) := \gamma(c) + \alpha(a)$, is also linear and continuous. The two maps are clearly inverses of each other. ■

Lemma 3.6 *If one has the exact sequence of Banach spaces*

$$0 \longrightarrow U \xrightarrow{\iota} V \xrightarrow{\pi} W \longrightarrow 0 \quad (3.3)$$

then the dual sequence

$$0 \longrightarrow W^* \xrightarrow{\pi^*} V^* \xrightarrow{\iota^*} U^* \longrightarrow 0 \quad (3.4)$$

is also exact.

Proof. The linear continuous map π^* is injective. Indeed, if $\gamma \in \ker \pi^* \subset W^*$, then $(\gamma \circ \pi)(v) = 0$ for all $v \in V$. Surjectivity of π implies then that $\gamma(w) = 0$ for all $w \in W$ so $\gamma = 0$.

The linear continuous map ι^* is surjective. Indeed, since $\iota(U) = \ker \pi$ is a closed subspace of V and ι is injective, $\iota : U \rightarrow \iota(U)$ is a Banach space isomorphism. Then, if $\alpha \in U^*$, the linear functional $\alpha \circ \iota^{-1} : \iota(U) \rightarrow \mathbb{C}$ is continuous. Extend this functional to $\beta \in V^*$ by the Hahn-Banach Theorem. Thus, for any $u \in U$, we have $\iota^*(\beta)(u) = \beta(\iota(u)) = (\alpha \circ \iota^{-1})(\iota(u)) = \alpha(u)$, which shows that $\iota^*(\beta) = \alpha$, that is, ι^* is onto.

Since $\iota^* \circ \pi^* = (\pi \circ \iota)^* = 0$ by exactness of the sequence (3.3), it follows that $\pi^*(W^*) \subset \ker \iota^*$. To prove the opposite inclusion, let $\beta \in V^*$ be such that $\iota^*(\beta) = 0$. Define $\tilde{\beta} : W \rightarrow \mathbb{C}$ by $\tilde{\beta}(w) = \beta(v)$, if $w = \pi(v)$; thus $\tilde{\beta} \circ \pi = \beta$. Since $\beta|_{\iota(U)} = 0$ by hypothesis and $\iota(U) = \ker \pi$ by exactness of (3.3), it follows that $\tilde{\beta}$ is well defined. It is straightforward to verify that $\tilde{\beta}$ is linear and continuous using the Banach space isomorphism $V/\iota(U) \cong W$. Finally, $\beta = \tilde{\beta} \circ \pi = \pi^*(\tilde{\beta}) \in \pi^*(W^*)$. ■

Lemma 3.7 *Assume that all Banach spaces in the exact sequence (3.3) admit preduals, that is, there are Banach spaces U_* , V_* , and W_* such that $U = (U_*)^*$, $V = (V_*)^*$, and $W = (W_*)^*$, respectively. Assume, in addition, that the dual maps π^* and ι^* preserve the predual spaces, that is, $\pi^*(W_*) \subset V_*$ and $\iota^*(V_*) \subset U_*$. Then one has the following commutative diagram of exact sequences*

$$\begin{array}{ccccccc}
0 & \longrightarrow & W^* & \xrightarrow{\pi^*} & V^* & \xrightarrow{\iota^*} & U^* \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & W_* & \xrightarrow{\pi_*} & V_* & \xrightarrow{\iota_*} & U_* \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & 0 & & 0 & & 0
\end{array}$$

where all vertical arrows are inclusions and the maps in the second line are defined by restriction, that is, $\pi_* := \pi^*|_{W_*}$ and $\iota_* := \iota^*|_{V_*}$.

In particular, if $V \cong U \oplus W$, $U = (U_*)^*$, $W = (W_*)^*$, and the maps ι and π in the sequence

$$0 \longrightarrow U \xrightarrow{\iota} U \oplus W \xrightarrow{\pi} W \longrightarrow 0$$

are defined by $\iota(u) := (u, 0)$, $\pi(u, w) := w$, for $u \in U$, $w \in W$, then the commutative diagram above is valid for $V_* = U_* \oplus W_*$.

Proof. Since $\pi^*(W_*) \subset V_*$ and $\iota^*(V_*) \subset U_*$, the maps in the second line are well defined. The map π_* is injective because it is the restriction of the injective map π^* to the Banach subspace $W_* \subset W^*$.

The relation $\pi \circ \iota = 0$ implies that $\iota_* \circ \pi_* = (\iota^* \circ \pi^*)|_{W_*} = (\pi \circ \iota)^*|_{W_*} = 0$ which shows that

$$\text{im } \pi_* \subset \ker \iota_*. \quad (3.5)$$

Let us assume that there is some element $b \in \ker \iota_*$ such that $b \notin \overline{\text{im } \pi_*} \subset V_*$. Then, by the Hahn-Banach Theorem, there exists an element $v \in V$ such that $v(b) \neq 0$ and $v|_{\overline{\text{im } \pi_*}} \equiv 0$. The exactness of the sequence (3.3) and $b \in \ker \iota_* \subset V_* \subset V^*$ implies that

$$b|_{\ker \pi} = 0. \quad (3.6)$$

However, one concludes from $v|_{\overline{\text{im } \pi_*}} \equiv 0$ that

$$0 = v(\pi_*(\gamma)) = \pi^*(\gamma)(v) = \gamma(\pi(v))$$

for any $\gamma \in W_*$, which implies that $\pi(v) = 0$, that is, $v \in \ker \pi$. By (3.6) we have $v(b) = 0$, which contradicts the choice of v . Thus we have proved that

$$\overline{\text{im } \pi_*} = \ker \iota_*.$$

Thus, for any $b \in \ker \iota_*$ there is a sequence $\{a_n\}_{n=1}^\infty \subset W_*$ such that $\pi^*(a_n) = \pi_*(a_n) \rightarrow b$ as $n \rightarrow \infty$ and, since $\text{im } \pi^* = \ker \iota^*$, there is some element $a \in W^*$ such that $\pi^*(a) = b$. One obtains from the above $\pi^*(a_n - a) \rightarrow 0$ as $n \rightarrow \infty$. Now, because $\pi^* : W^* \rightarrow \pi^*(W^*) = \ker \iota^*$ is a continuous isomorphism between Banach spaces, the Banach Isomorphism Theorem guarantees that its inverse is also continuous and thus we have $a_n \rightarrow a$ as $n \rightarrow \infty$. Since W_* is closed in W^* , this implies that $a \in W_*$ and therefore $b \in \text{im } \pi_*$. This shows that

$$\text{im } \pi_* = \ker \iota_*. \quad (3.7)$$

The last step of the proof is to show that $\text{im } \iota_* = U_*$. If $\overline{\text{im } \iota_*} \neq U_*$, then there are elements $c \notin \overline{\text{im } \iota_*}$ and $u \in U$ such that $u(c) \neq 0$ and $u|_{\overline{\text{im } \iota_*}} \equiv 0$. Thus one has $0 = u(\iota_*(b)) = b(\iota(u))$ for any $b \in V_*$, which means that $\iota(u) = 0$. Injectivity of ι implies then that $u = 0$, which contradicts the choice $u(c) \neq 0$. Thus we showed that

$$\overline{\text{im } \iota_*} = U_*.$$

Therefore, for any $c \in U_*$ there is a sequence $\{b_n\}_{n=1}^\infty \subset V_*$ and an element $b \in V^*$ such that $\iota_*(b_n) \rightarrow c = \iota^*(b)$ as $n \rightarrow \infty$. Thus $\iota^*(b_n - b) \rightarrow 0$ as $n \rightarrow \infty$ and, using the isomorphism $V^*/\pi^*(W^*) \cong U^*$, we conclude from the Banach Isomorphism Theorem that $[b_n - b] \rightarrow 0$ as $n \rightarrow \infty$ in $V^*/\pi^*(W^*)$, where $[a]$ denotes the equivalence class of $a \in V^*$ in the quotient Banach space $V^*/\pi^*(W^*)$. This means that there is a subsequence $\{b_{n_k}\}_{k=1}^\infty \subset V_*$ and an element $d \in W^*$ such that $b_{n_k} - b \rightarrow \pi^*(d)$ as $k \rightarrow \infty$. Therefore, since V_* is closed in V^* , we have that $\lim_{k \rightarrow \infty} b_{n_k} \in V_*$ and one has

$$c = \iota^*(b) = \iota^*(\lim_{k \rightarrow \infty} b_{n_k} - \pi^*(d)) = \iota^*(\lim_{k \rightarrow \infty} b_{n_k})$$

since $\iota^* \circ \pi^* = 0$. This shows that $c \in \text{im } \iota_*$ which proves that $\text{im } \iota_* = U_*$.

To prove the last statement, notice that by Lemma 3.5 we have $(U_* \oplus W_*)^* \cong (U_*)^* \oplus (W_*)^* \cong U \oplus W \cong V$, that is, $U_* \oplus W_*$ is a predual of V . In addition, it is easy to see that if $f \in W_* \subset W^*$ and $(g_1, g_2) \in U_* \oplus W_* \subset U^* \oplus W^*$, then $\pi^*(f) = (0, f) \in U_* \oplus W_*$ and $\iota^*(g_1, g_2) = g_1 \in U_*$ and thus the hypotheses in the first part of the lemma are verified. \blacksquare

We have now the necessary background to prove the main theorem of this section.

Proof of Theorem 3.4. Let us assume that \mathfrak{a} , \mathfrak{b} , and \mathfrak{c} form an exact sequence of Banach Lie-Poisson spaces in the sense of Definition 3.3. Lemma 3.6 guarantees that their duals also form an exact sequence

$$0 \longrightarrow \mathfrak{c}^* \xrightarrow{p^*} \mathfrak{b}^* \xrightarrow{j^*} \mathfrak{a}^* \longrightarrow 0$$

of Banach spaces. From Propositions 2.4 and 2.5 it follows that this sequence is an exact sequence of Banach Lie algebras in the category \mathfrak{L}_0 .

Conversely, assume that \mathfrak{n} , \mathfrak{g} , and \mathfrak{h} form an exact sequence of Banach Lie algebras in the category \mathfrak{L}_0 (see Definition 3.2). Thus they have preduals \mathfrak{n}_* , \mathfrak{g}_* , and \mathfrak{h}_* which, by Lemma 3.7, form an exact sequence

$$0 \longrightarrow \mathfrak{h}_* \xrightarrow{\pi_*} \mathfrak{g}_* \xrightarrow{\iota_*} \mathfrak{n}_* \longrightarrow 0$$

of Banach spaces. Now, again by Propositions 2.4, 2.5, and Theorem 2.1, the preduals \mathfrak{n}_* , \mathfrak{g}_* , and \mathfrak{h}_* are Banach Lie Poisson spaces and the maps π_* and ι_* are linear Poisson maps.

The last statement of the theorem is proved in the following way. By the second part of Lemma 3.7, $\mathfrak{a} \oplus \mathfrak{c}$ can be taken as the predual space to $\mathfrak{h} \oplus \mathfrak{n}$. By the first part of the theorem, the natural maps j and p are linear Poisson maps. The desired conclusion now immediately follows from Theorem 2.6, if we show that the spaces $\pi_{\mathfrak{a}}^*(C^\infty(\mathfrak{a}))$ and $\pi_{\mathfrak{c}}^*(C^\infty(\mathfrak{c}))$ are Poisson commuting subalgebras of $C^\infty(\mathfrak{a} \oplus \mathfrak{c})$, where $\pi_{\mathfrak{a}} : \mathfrak{a} \oplus \mathfrak{c} \rightarrow \mathfrak{a}$ and $\pi_{\mathfrak{c}} : \mathfrak{a} \oplus \mathfrak{c} \rightarrow \mathfrak{c}$ are the natural projections. If $f \in C^\infty(\mathfrak{a})$ and $g \in C^\infty(\mathfrak{c})$, then $Df(a) \in \mathfrak{a}^* = \mathfrak{h}$ and $Dg(c) \in \mathfrak{c}^* = \mathfrak{n}$, so that $[D(\pi_{\mathfrak{a}}^* f)(a, c), D(\pi_{\mathfrak{c}}^* g)(a, c)] = [(Df(a), 0), (0, Dg(c))] = 0$, since $\mathfrak{h} \oplus \mathfrak{n}$ is a direct sum of Lie algebras. Formula (2.1) of the Lie-Poisson bracket insures then that $\{\pi_{\mathfrak{a}}^* f, \pi_{\mathfrak{c}}^* g\} = 0$ as required. ■

Therefore, one concludes from Theorem 3.4 that the problem of extensions in the category of Banach Lie-Poisson spaces is equivalent to that in the category \mathfrak{L}_0 .

It was shown in Odziejewicz and Ratiu [2003] that on the predual Banach space of a W^* -algebra (von Neumann algebra) there is a canonically defined Banach Lie-Poisson structure. Since the theory of von Neumann algebras is closely related to crucial problems of quantum physics (Emch [1972], Bratteli [1979], [1981]) we shall discuss the application of Theorem 3.4 to this subcase.

Recall that a W^* -algebra is a C^* -algebra \mathfrak{m} which posses a predual Banach space \mathfrak{m}_* , i.e. $\mathfrak{m} = (\mathfrak{m}_*)^*$; this predual is unique (Sakai [1971]). Since $\mathfrak{m}^* = (\mathfrak{m}_*)^{**}$, the predual Banach space \mathfrak{m}_* canonically embeds into the Banach space \mathfrak{m}^* dual to \mathfrak{m} . Thus we shall always think of \mathfrak{m}_* as a Banach subspace of \mathfrak{m}^* . The existence of \mathfrak{m}_* allows the introduction of the $\sigma(\mathfrak{m}, \mathfrak{m}_*)$ -topology on the W^* -algebra \mathfrak{m} ; for simplicity we shall call it the σ -topology in the sequel. Recall that a net $\{x_\alpha\}_{\alpha \in A} \subset \mathfrak{m}$ converges to $x \in \mathfrak{m}$ in the σ -topology if, by definition, $\lim_{\alpha \in A} \langle x_\alpha, b \rangle = \langle x, b \rangle$ for all $b \in \mathfrak{m}_*$. The predual space \mathfrak{m}_* is characterized as the subspace of \mathfrak{m}^* consisting of all σ -continuous linear functionals on \mathfrak{m} (Sakai [1971]).

A **homomorphism of W^* -algebras** $\phi : \mathfrak{m}_1 \rightarrow \mathfrak{m}_2$ is a σ -continuous $*$ -algebra homomorphism (and is hence automatically norm continuous; Sakai [1971]). Note that $\phi^*(\mathfrak{m}_{2*}) \subset \mathfrak{m}_{1*}$, where \mathfrak{m}_{i*} is the unique predual of \mathfrak{m}_i , for $i = 1, 2$. Indeed, since any element $b \in \mathfrak{m}_{2*}$ is σ -continuous on \mathfrak{m}_2 and ϕ is also σ -continuous, it follows that $\phi^*(b) = b \circ \phi$ is σ -continuous on \mathfrak{m}_1 and hence is an element of \mathfrak{m}_{1*} .

Conversely, assume that \mathfrak{m}_1 and \mathfrak{m}_2 are W^* -algebras and that $\phi : \mathfrak{m}_1 \rightarrow \mathfrak{m}_2$ is a $*$ -homomorphism satisfying $\phi^*(\mathfrak{m}_{2*}) \subset \mathfrak{m}_{1*}$. Then ϕ is σ -continuous. Indeed, if $\{x_\alpha\}_{\alpha \in A} \subset \mathfrak{m}_1$ is a net σ -converging to $x \in \mathfrak{m}_1$, then for any $b_1 \in \mathfrak{m}_{1*}$ we have $\lim_{\alpha \in A} \langle x_\alpha, b_1 \rangle = \langle x, b_1 \rangle$. We have for any $b_2 \in \mathfrak{m}_{2*}$, $\lim_{\alpha \in A} \langle \phi(x_\alpha), b_2 \rangle = \lim_{\alpha \in A} \langle x_\alpha, b_2 \circ \phi \rangle = \langle x, b_2 \circ \phi \rangle$, since, by hypothesis, $b_2 \circ \phi = \phi^*(b_2) \in \mathfrak{m}_{1*}$. This shows that $\lim_{\alpha \in A} \langle \phi(x_\alpha), b_2 \rangle = \langle \phi(x), b_2 \rangle$ for any $b_2 \in \mathfrak{m}_{2*}$, that is, $\phi : \mathfrak{m}_1 \rightarrow \mathfrak{m}_2$ is σ -continuous. These arguments prove the following.

Proposition 3.8 *Let \mathfrak{m}_1 and \mathfrak{m}_2 be W^* -algebras and $\phi : \mathfrak{m}_1 \rightarrow \mathfrak{m}_2$ a $*$ -homomorphism. Then ϕ is a W^* -algebra homomorphism if and only if ϕ^* preserves the preduals, that is, $\phi^*(\mathfrak{m}_{2*}) \subset \mathfrak{m}_{1*}$.*

Since any W^* -algebra is a Banach Lie algebra relative to the commutator bracket and possesses a unique predual, this proposition plus the condition $\text{ad}_a^* \mathfrak{m}_* \subset \mathfrak{m}_* \subset \mathfrak{m}^*$ for any $a \in \mathfrak{m}$, shows that \mathfrak{W} is a subcategory of \mathfrak{L}_{0u} (see Theorem 2.1). The condition $\text{ad}_a^* \mathfrak{m}_* \subset \mathfrak{m}_*$ for all $a \in \mathfrak{m}_*$ is always satisfied. Indeed, left and right multiplication by $a \in \mathfrak{m}$ define uniformly and σ -continuous maps $L_a : \mathfrak{m} \ni x \mapsto ax \in \mathfrak{m}$ and $R_a : \mathfrak{m} \ni x \mapsto xa \in \mathfrak{m}$ (Sakai [1971]). Let $L_a^* : \mathfrak{m}^* \rightarrow \mathfrak{m}^*$ and $R_a^* : \mathfrak{m}^* \rightarrow \mathfrak{m}^*$ denote the dual maps of L_a and R_a respectively. If $v \in \mathfrak{m}_*$, then $L_a^*(v)$ and $R_a^*(v)$ are σ -continuous functionals and therefore, by the characterization of the predual \mathfrak{m}_* as the subspace of σ -continuous functionals in \mathfrak{m}^* , it follows that $L_a^*(v), R_a^*(v) \in \mathfrak{m}_*$. Since $\text{ad}_a = [a, \cdot] = L_a - R_a$ it follows that $\text{ad}_a^* = L_a^* - R_a^*$ and hence for any $v \in \mathfrak{m}_*$, we have that $\text{ad}_a^*(v) = L_a^*(v) - R_a^*(v) \in \mathfrak{m}_*$.

This shows that \mathfrak{m}_* is a Banach Lie-Poisson space with the Poisson bracket $\{f, g\}$ of $f, g \in C^\infty(\mathfrak{m}_*)$ given by (2.1). The Hamiltonian vector field X_f defined by the smooth function $f \in C^\infty(\mathfrak{m}_*)$ is given by (2.2).

An **exact sequence of W^* -algebras** is an exact sequence of algebras in which all maps are W^* -homomorphisms.

Let us analyze exact sequences of Banach Lie-Poisson spaces that are preduals of W^* -algebras. So assume that (3.2) is an exact sequence of Banach Lie-Poisson spaces such that their duals $\mathfrak{n} := \mathfrak{c}^*$, $\mathfrak{g} := \mathfrak{b}^*$, $\mathfrak{h} := \mathfrak{a}^*$ are W^* -algebras. By Theorem 3.4, the sequence (3.1), where $\iota := p^*$ and $\pi := j^*$, is an exact sequence of Banach Lie algebras in the category \mathfrak{L}_0 but all objects in the sequence are W^* -algebras. This means that the maps ι and π are Banach Lie algebra homomorphisms and that $\iota^*(\mathfrak{b}) \subset \mathfrak{c}$, $\pi^*(\mathfrak{a}) \subset \mathfrak{b}$. What is not guaranteed, and is not true in general, is that the linear continuous maps ι and π are homomorphisms of the associative product structures of the W^* -algebras $\mathfrak{n}, \mathfrak{g}, \mathfrak{h}$. An example of a Lie algebra homomorphism between W^* -algebras that is not a homomorphism for the associative product is given by $\phi : \text{gl}(n) \rightarrow \text{gl}(n)$, $\phi(a) := \text{tr}(a)\mathbb{I}$, where $\text{gl}(n)$ is the algebra of $n \times n$ matrices and \mathbb{I} the identity matrix.

We shall assume now that ι and π are $*$ -homomorphisms for the associative product structure. Then, since they preserve the preduals, Proposition 3.8 insures that they are W^* -homomorphisms. Conversely, assume that (3.1) is an exact sequence of W^* -algebras. Then the maps j and p are homomorphisms of Banach Lie algebras and, by Proposition 3.8, their duals preserve the predual spaces, that is, this is an exact sequence in the category \mathfrak{L}_{0u} .

Thus we are lead to consider exact sequences (3.1) of W^* -algebras. Then $\ker \pi = \text{im } \iota$ is a σ -closed ideal in \mathfrak{g} and thus there exist a central projector $z \in \mathfrak{g}$ such that $\text{im } \iota = z\mathfrak{g}$ (Proposition 1.10.5 in Sakai [1971]). The projector $1 - z$ is also central so that denoting $(\text{im } \iota)^\perp := (1 - z)\mathfrak{g}$, one has the direct sum splitting

$$\mathfrak{g} = \text{im } \iota \oplus (\text{im } \iota)^\perp$$

into two σ -closed ideals of \mathfrak{g} . It is easy to see that for any $x \in \text{im } \iota$ and $y \in (\text{im } \iota)^\perp$ we have $xy = 0$. Thus the W^* -algebra \mathfrak{g} is the direct sum of two commuting σ -closed ideals. In addition, the map $\pi|_{(\text{im } \iota)^\perp} : (\text{im } \iota)^\perp \rightarrow \mathfrak{h}$ is an isomorphism of W^* -algebras. This proves the following.

Proposition 3.9 *Any extension*

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0$$

of the W^* -algebra \mathfrak{h} by the W^* -algebra \mathfrak{n} (where ι and π are $*$ -homomorphisms of the associative product structure) is isomorphic to the extension

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathfrak{n} \oplus \mathfrak{h} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0,$$

where $\iota(n) := (n, 0)$ and $\pi(n, h) := h$, for $n \in \mathfrak{n}$ and $h \in \mathfrak{h}$.

This proposition together with Theorems 3.4 and 2.6 immediately yields the following result.

Proposition 3.10 *Any extension*

$$0 \longrightarrow \mathfrak{a} \xrightarrow{j} \mathfrak{b} \xrightarrow{p} \mathfrak{c} \longrightarrow 0$$

of the Banach Lie-Poisson space \mathfrak{c} by the Banach Lie-Poisson space \mathfrak{a} such that the dual sequence is an exact sequence of W^* -algebras, is isomorphic to the extension

$$0 \longrightarrow \mathfrak{a} \xrightarrow{j} \mathfrak{a} \oplus \mathfrak{c} \xrightarrow{p} \mathfrak{c} \longrightarrow 0,$$

where $j(a) := (a, 0)$ and $\pi(a, c) = c$, for $a \in \mathfrak{a}$ and $c \in \mathfrak{c}$. This means that \mathfrak{b} is Poisson isomorphic to the product of the Banach Lie-Poisson spaces \mathfrak{a} and \mathfrak{c} in the sense of Theorem 2.6.

4 Extensions of Banach Lie algebras

As we showed in the previous section, the problem of extension of Banach Lie-Poisson spaces reduces to the problem of extensions of Banach Lie algebras in the subcategory \mathfrak{L}_0 . We begin with some general considerations in the category of Banach Lie algebras.

Let $\text{aut}(\mathfrak{n}) := \{D : \mathfrak{n} \rightarrow \mathfrak{n} \mid D \text{ derivation of } \mathfrak{n}\}$ be the Banach Lie algebra of all continuous linear derivations of \mathfrak{n} . Recall that $D : \mathfrak{n} \rightarrow \mathfrak{n}$ is a **derivation** if $D[\eta, \zeta] = [D\eta, \zeta] + [\eta, D\zeta]$ for all $\eta, \zeta \in \mathfrak{n}$. Denote by $\text{int}(\mathfrak{n}) := \{\text{ad}_\eta \mid \eta \in \mathfrak{n}\}$ the subalgebra of $\text{aut}(\mathfrak{n})$ consisting of **inner derivations**. In general, this is not a closed subspace of $\text{aut}(\mathfrak{n})$. *In this section we shall assume that $\text{int}(\mathfrak{n})$ is closed in $\text{aut}(\mathfrak{n})$ and hence $\text{int}(\mathfrak{n})$ is then a Banach Lie ideal of $\text{aut}(\mathfrak{n})$.* Denote by $\text{out}(\mathfrak{n}) := \text{aut}(\mathfrak{n})/\text{int}(\mathfrak{n})$, the Banach Lie algebra of **outer derivations** of \mathfrak{n} . The norm on $\text{int}(\mathfrak{n})$ and $\text{aut}(\mathfrak{n})$ is the usual operator norm induced from the space $\text{gl}(\mathfrak{n})$ of all linear continuous maps of \mathfrak{n} into itself. The norm on $\text{out}(\mathfrak{n})$ is the quotient norm.

Let $C_{\mathfrak{g}}(\mathfrak{n}) := \{\xi \in \mathfrak{g} \mid [\xi, \zeta] = 0 \text{ for all } \zeta \in \mathfrak{n}\}$ be the **centralizer** of \mathfrak{n} in \mathfrak{g} and $C(\mathfrak{n}) := \{\eta \in \mathfrak{n} \mid [\eta, \zeta] = 0 \text{ for all } \zeta \in \mathfrak{n}\} = C_{\mathfrak{g}}(\mathfrak{n}) \cap \mathfrak{n}$ be the **center** of \mathfrak{n} ; $C(\mathfrak{n})$ is a closed ideal in $C_{\mathfrak{g}}(\mathfrak{n})$ which is itself a Banach Lie subalgebra of \mathfrak{g} . Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & C(\mathfrak{n}) & \longrightarrow & C_{\mathfrak{g}}(\mathfrak{n}) & \longrightarrow & C_{\mathfrak{g}}(\mathfrak{n})/C(\mathfrak{n}) \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathfrak{n} & \xrightarrow{\iota} & \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{h} \longrightarrow 0 \\
& \downarrow \text{ad} & & \downarrow \text{ad}|_{\mathfrak{n}} & & \downarrow \sigma & \\
0 & \longrightarrow & \text{int}(\mathfrak{n}) & \longrightarrow & \text{aut}(\mathfrak{n}) & \longrightarrow & \text{out}(\mathfrak{n}) \longrightarrow 0 \\
& \downarrow & & & & & \\
& 0 & & & & &
\end{array}$$

All maps that are not labeled are natural: they are inclusions or projections on quotients. In particular, \mathfrak{h} is isomorphic as a Banach Lie algebra with the quotient $\mathfrak{g}/\mathfrak{n}$ endowed with the quotient norm. The map $\text{ad} : \mathfrak{n} \rightarrow \text{int}(\mathfrak{n})$ is given by $\eta \mapsto \text{ad}_\eta := [\eta, \cdot]$, for $\eta \in \mathfrak{n}$. The map $\text{ad}|_{\mathfrak{n}} : \mathfrak{g} \rightarrow \text{aut}(\mathfrak{g})$ is given by $\xi \mapsto \text{ad}_\xi|_{\mathfrak{n}}$ for $\xi \in \mathfrak{g}$.

Finally, the map $\sigma : \mathfrak{h} \rightarrow \text{out}(\mathfrak{h})$ is defined in the following way. Let $\eta \in \mathfrak{h}$ and $\xi \in \pi^{-1}(\eta) \subset \mathfrak{g}$. Define $\sigma(\eta) := [\text{ad}_\xi|_{\mathfrak{n}}] \in \text{out}(\mathfrak{h})$, where $[\text{ad}_\xi|_{\mathfrak{n}}]$ denotes the equivalence class of the operator $\text{ad}_\xi|_{\mathfrak{n}} \in \text{aut}(\mathfrak{g})$ with respect to the ideal $\text{int}(\mathfrak{n})$. The map σ is well defined for if $\xi_1, \xi_2 \in \pi^{-1}(\eta)$, then exactness of (3.1) implies that $\xi_1 - \xi_2 \in \mathfrak{n}$ and thus $\text{ad}_{\xi_1 - \xi_2}|_{\mathfrak{n}} \in \text{int}(\mathfrak{n})$. Therefore, $[\text{ad}_{\xi_1}|_{\mathfrak{n}}] = [\text{ad}_{\xi_2}|_{\mathfrak{n}}]$. The map σ is clearly a Lie algebra homomorphism. The defining equality for σ , that is, $(\sigma \circ \pi)(\xi) = [\text{ad}_\xi|_{\mathfrak{n}}]$ for every $\xi \in \mathfrak{g}$, proves that the lower right square of the diagram is commutative. Continuity of σ is proved in the following way. Since \mathfrak{g} is a Banach Lie algebra, there is a constant $C > 0$ such that $\|[\xi, \zeta]\| \leq C\|\xi\|\|\zeta\|$, for all $\xi, \zeta \in \mathfrak{g}$. Therefore, $\|\text{ad}_\xi\| \leq C\|\xi\|$, for all $\xi \in \mathfrak{g}$. Thus, for $\eta \in \mathfrak{h}$ and $\xi \in \pi^{-1}(\eta)$ arbitrary, we have $\|\sigma(\eta)\| = \|[\text{ad}_\xi|_{\mathfrak{n}}]\| \leq \|\text{ad}_\xi|_{\mathfrak{n}}\| \leq \|\text{ad}_\xi\| \leq C\|\xi\|$, which shows, using the isomorphism $\mathfrak{g}/\mathfrak{n} \cong \mathfrak{h}$ and the definition of the norm on the quotient, that σ is continuous.

Note that $\text{ad}|_{\mathfrak{n}} : \mathfrak{g} \rightarrow \text{aut}(\mathfrak{g})$ and $\sigma : \mathfrak{h} \rightarrow \text{out}(\mathfrak{h})$ are not surjective, in general.

Consider now a linear continuous section $s : \mathfrak{h} \rightarrow \mathfrak{g}$, $\pi \circ s = \text{id}_{\mathfrak{h}}$, and assume that $\mathfrak{g} = \mathfrak{n} \oplus s(\mathfrak{h})$, that is, $s(\mathfrak{h})$ is closed and has as split complement the space \mathfrak{n} . Define the isomorphism of Banach spaces $\psi : \mathfrak{g} \rightarrow \mathfrak{n} \oplus \mathfrak{h}$ by $\psi(\xi) = (\xi - s(\pi(\xi)), \pi(\xi))$, whose inverse is given by $\psi^{-1}(\zeta, \eta) := \zeta + s(\eta)$, for $\eta \in \mathfrak{h}$, $\zeta \in \mathfrak{n}$, and $\xi \in \mathfrak{g}$. Note that $\psi^{-1}(0, \eta) = s(\eta)$ for any $\eta \in \mathfrak{h}$ and $\psi^{-1}(\zeta, 0) = \zeta$ for any $\zeta \in \mathfrak{n}$. Conversely, given an isomorphism of Banach spaces $\psi : \mathfrak{g} \rightarrow \mathfrak{n} \oplus \mathfrak{h}$ that is the identity on \mathfrak{n} , it determines a linear continuous section $s : \mathfrak{h} \rightarrow \mathfrak{g}$ by $s(\eta) := \psi^{-1}(0, \eta)$ whose image $s(\mathfrak{h}) = \psi^{-1}(\mathfrak{h})$ is a closed split subspace of \mathfrak{g} admitting \mathfrak{n} as a complement. Thus *there is a bijective correspondence*

between the Banach space isomorphisms $\psi : \mathfrak{g} \rightarrow \mathfrak{n} \oplus \mathfrak{h}$ that are the identity on \mathfrak{n} and the linear continuous sections $s : \mathfrak{h} \rightarrow \mathfrak{g}$ with closed split range admitting \mathfrak{n} as a complement.

From this point on we shall assume that \mathfrak{g} is isomorphic to $\mathfrak{n} \oplus \mathfrak{h}$ as a Banach space. Let us stress that this sum is *not* taken, in general, as a direct sum of Banach Lie algebras. The isomorphism ψ induces a Lie bracket on $\mathfrak{n} \oplus \mathfrak{h}$ by

$$\begin{aligned} [(\zeta, \eta), (\zeta', \eta')] &:= \psi([\psi^{-1}(\zeta, \eta), \psi^{-1}(\zeta', \eta')]) \\ &= ([\zeta, \zeta'] + \varphi(\eta)(\zeta') - \varphi(\eta')(\zeta) + \omega(\eta, \eta'), [\eta, \eta']), \end{aligned} \quad (4.1)$$

where $\omega : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{n}$ and $\varphi : \mathfrak{h} \rightarrow \text{aut}(\mathfrak{n})$ are defined by

$$\omega(\eta, \eta') := [s(\eta), s(\eta')] - s([\eta, \eta']) \quad (4.2)$$

$$\varphi(\eta) := [s(\eta), \cdot] \quad (4.3)$$

for any $\zeta, \zeta' \in \mathfrak{n}$ and $\eta, \eta' \in \mathfrak{h}$.

Let us pose the inverse question. Given are two Banach Lie algebras \mathfrak{n} and \mathfrak{h} . Endow the direct sum Banach space $\mathfrak{n} \oplus \mathfrak{h}$ with the continuous bilinear skew symmetric operation given by (4.1), where $\omega : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{n}$ is a given continuous bilinear skew symmetric map and $\varphi : \mathfrak{h} \rightarrow \text{aut}(\mathfrak{n})$ is a given continuous linear map. What are the conditions on ω and φ so that this operation $[(\zeta, \eta), (\zeta', \eta')]$ defined by the right hand side of (4.1) is a Lie bracket on $\mathfrak{n} \oplus \mathfrak{h}$? The answer to this question is given by the following proposition.

Proposition 4.1 *Let \mathfrak{n} and \mathfrak{h} be two Banach Lie algebras, $\omega : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{n}$ a continuous bilinear skew symmetric map, and $\varphi : \mathfrak{h} \rightarrow \text{aut}(\mathfrak{n})$ a continuous linear map. Then*

$$[(\zeta, \eta), (\zeta', \eta')] = ([\zeta, \zeta'] + \varphi(\eta)(\zeta') - \varphi(\eta')(\zeta) + \omega(\eta, \eta'), [\eta, \eta']) \quad (4.4)$$

for $\zeta \in \mathfrak{n}$ and $\eta \in \mathfrak{h}$ endows the Banach space direct sum $\mathfrak{g} := \mathfrak{n} \oplus \mathfrak{h}$ with a Banach Lie algebra structure if and only if

$$\begin{aligned} &\omega([\eta, \eta'], \eta'') + \omega([\eta', \eta''], \eta) + \omega([\eta'', \eta'], \eta') \\ &- \varphi(\eta)(\omega(\eta', \eta'')) - \varphi(\eta')(\omega(\eta'', \eta)) - \varphi(\eta'')(\omega(\eta, \eta')) = 0 \end{aligned} \quad (4.5)$$

and

$$\text{ad}_{\omega(\eta, \eta')} + \varphi([\eta, \eta']) - [\varphi(\eta), \varphi(\eta')] = 0 \quad (4.6)$$

for any $\eta, \eta', \eta'' \in \mathfrak{h}$. Consequently, the Banach Lie algebra $\mathfrak{n} \oplus \mathfrak{h}$ is an extension of the Banach Lie algebra \mathfrak{n} by the Banach Lie algebra \mathfrak{h} .

Proof. Since φ is linear continuous and ω is bilinear continuous, the bilinear skew symmetric operation defined in (4.4) is also continuous. So it is enough to show that (4.5) and (4.6) are equivalent to the Jacobi identity.

From the expression of the second component in (4.4), it follows that the Jacobi identity gives no conditions on it since \mathfrak{h} is a Lie algebra. Thus only the first components need to be calculated. A direct computation shows that the first component of

$[[(\zeta, \eta), (\zeta', \eta')], (\zeta'', \eta'')] equals$

$$\begin{aligned} & [[\zeta, \zeta'], \zeta''] + [\varphi(\eta)(\zeta'), \zeta''] - [\varphi(\eta')(\zeta), \zeta''] + [\omega(\eta, \eta'), \zeta''] \\ & + \varphi([\eta, \eta'])(\zeta'') - \varphi(\eta'')([\zeta, \zeta']) - \varphi(\eta'')\varphi(\eta)(\zeta') \\ & + \varphi(\eta'')\varphi(\eta')(\zeta) - \varphi(\eta'')(\omega(\eta, \eta')) + \omega([\eta, \eta'], \eta''). \end{aligned} \quad (4.7)$$

For $\zeta = \zeta' = \zeta'' = 0$ this expression becomes $\omega([\eta, \eta'], \eta'') - \varphi(\eta'')(\omega(\eta, \eta'))$. Taking the circular permutations of (4.7) and then setting $\zeta = \zeta' = \zeta'' = 0$ yields

$$\begin{aligned} & \omega([\eta, \eta'], \eta'') + \omega([\eta', \eta''], \eta) + \omega([\eta'', \eta'], \eta') \\ & - \varphi(\eta)(\omega(\eta', \eta'')) - \varphi(\eta')(\omega(\eta'', \eta)) - \varphi(\eta'')(\omega(\eta, \eta')) = 0 \end{aligned}$$

for any $\eta, \eta', \eta'' \in \mathfrak{h}$. This proves (4.5).

In the sum of (4.7) with the two terms obtained from it by circular permutations, there are expressions that add up to zero. The sum of the first term in (4.7) with its circular permutations is zero since it is the Jacobi identity in the Lie algebra \mathfrak{n} . By (4.5), the sum of the ninth and the tenth term in (4.7) plus their circular permutations add also up to zero. Thus, the sum of (4.7) with its circular permutations equals

$$\begin{aligned} & [\omega(\eta, \eta'), \zeta''] + [\omega(\eta', \eta''), \zeta] + [\omega(\eta'', \eta), \zeta'] \\ & + \varphi([\eta, \eta'])(\zeta'') + \varphi([\eta', \eta''])(\zeta) + \varphi([\eta'', \eta])(\zeta') \\ & - [\varphi(\eta), \varphi(\eta')](\zeta'') - [\varphi(\eta'), \varphi(\eta'')](\zeta) - [\varphi(\eta''), \varphi(\eta)](\zeta') \\ & + [\varphi(\eta)(\zeta'), \zeta''] - [\varphi(\eta)(\zeta''), \zeta'] - \varphi(\eta)([\zeta', \zeta'']) \\ & + [\varphi(\eta')(\zeta''), \zeta] - [\varphi(\eta')(\zeta), \zeta''] - \varphi(\eta')([\zeta'', \zeta]) \\ & + [\varphi(\eta'')(\zeta), \zeta'] - [\varphi(\eta'')(\zeta'), \zeta] - \varphi(\eta'')([\zeta, \zeta']). \end{aligned}$$

Each of the last three lines vanishes because $\varphi(\eta)$, $\varphi(\eta')$, and $\varphi(\eta'')$ are derivations on \mathfrak{n} . Since ζ , ζ' , and ζ'' are arbitrary, from the remaining three top lines we conclude that

$$\text{ad}_{\omega(\eta, \eta')} + \varphi([\eta, \eta']) - [\varphi(\eta), \varphi(\eta')] = 0$$

for any $\eta, \eta', \eta'' \in \mathfrak{h}$, which proves (4.6).

Conversely, suppose that $\omega : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{n}$ is a continuous bilinear skew symmetric map and $\varphi : \mathfrak{h} \rightarrow \text{aut}(\mathfrak{n})$ is a continuous linear map satisfying (4.5) and (4.6). A direct verification using (4.5) and (4.6) shows that the Jacobi identity holds. Thus (4.4) endows $\mathfrak{g} \oplus \mathfrak{h}$ with a Banach Lie algebra structure. ■

If \mathfrak{n} is Abelian, then $\text{ad}_{\omega(\eta, \eta')} = 0$ and $\text{aut}(\mathfrak{n}) = \text{gl}(\mathfrak{n})$, the Banach space of all linear continuous maps from \mathfrak{n} to \mathfrak{n} . Thus the second condition becomes $\varphi([\eta, \eta']) = [\varphi(\eta), \varphi(\eta')]$ for all $\eta \in \mathfrak{n}$, that is, $\varphi : \mathfrak{h} \rightarrow \text{gl}(\mathfrak{n})$ is a representation. The first condition asserts that ω is a \mathfrak{h} -cocycle relative to the representation φ .

We want to mention that such extensions of Lie algebras were considered in the purely algebraic context by Mori [1953], Shukla [1966], and, more recently, by Alekseevsky, Michor, and Ruppert [2000].

5 Extensions of Banach Lie-Poisson spaces

Let us take Banach Lie-Poisson spaces \mathfrak{a} and \mathfrak{c} and construct the extension \mathfrak{b} of \mathfrak{c} by \mathfrak{a} in the sense of the category \mathfrak{B} of Banach Lie-Poisson spaces. We restrict our considerations to the case \mathfrak{b} equals the direct sum $\mathfrak{c} \oplus \mathfrak{a}$ of Banach spaces. Thus one has the Banach space exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{j} \mathfrak{c} \oplus \mathfrak{a} \xrightarrow{p} \mathfrak{c} \longrightarrow 0, \quad (5.1)$$

where $j(a) := (0, a)$ and $p(c, a) := c$. The dual of this sequence is

$$0 \longrightarrow \mathfrak{n} \xrightarrow{p^*} \mathfrak{n} \oplus \mathfrak{h} \xrightarrow{j^*} \mathfrak{h} \longrightarrow 0, \quad (5.2)$$

where $\mathfrak{h} := \mathfrak{a}^*$, $\mathfrak{n} = \mathfrak{c}^*$, $p^*(\zeta) := (\zeta, 0)$ and $j^*(\zeta, \eta) = \eta$. Since \mathfrak{a} and \mathfrak{c} are Banach Lie-Poisson spaces we have

$$\text{ad}_{\mathfrak{h}}^* \mathfrak{a} \subset \mathfrak{a} \quad \text{and} \quad \text{ad}_{\mathfrak{n}}^* \mathfrak{c} \subset \mathfrak{c}. \quad (5.3)$$

By Theorem 3.4, the question whether (5.1) is an exact sequence of Banach Lie-Poisson spaces is equivalent to the question whether (5.2) is an exact sequence in the subcategory $\mathfrak{L}_0 \subset \mathfrak{L}$. Proposition 4.1 gives a necessary and sufficient condition for (5.2) to be an exact sequence in the category \mathfrak{L} . The same sequence is exact in the subcategory \mathfrak{L}_0 if and only if $\text{ad}_{\mathfrak{n} \oplus \mathfrak{h}}^*(\mathfrak{c} \oplus \mathfrak{a}) \subset \mathfrak{c} \oplus \mathfrak{a}$. In order to see what this means we use formula (4.4) to compute the coadjoint representation on $\mathfrak{n}^* \oplus \mathfrak{h}^*$ and get

$$\text{ad}_{(\zeta, \eta)}^*(c, a) = (\text{ad}_{\zeta}^* c + \varphi(\eta)^* c, \omega(\eta, \cdot)^* c - (\varphi(\cdot)\zeta)^* c + \text{ad}_{\eta}^* a) \quad (5.4)$$

for $c \in \mathfrak{n}^*$, $a \in \mathfrak{h}^*$, $\zeta \in \mathfrak{n}$, and $\eta \in \mathfrak{h}$. The requirement that this action preserves the preduals, together with the properties (5.3) implies that $\varphi(\eta)^*(\mathfrak{c}) \subset \mathfrak{c}$ and that $(\omega(\eta, \cdot)^* - (\varphi(\cdot)\zeta)^*)(\mathfrak{c}) \subset \mathfrak{a}$ for all $\eta \in \mathfrak{h}$ and all $\zeta \in \mathfrak{n}$. Taking here alternatively $\eta = 0$ and $\zeta = 0$, the second condition becomes $\omega(\eta, \cdot)^*(\mathfrak{c}) \subset \mathfrak{a}$ and $(\varphi(\cdot)\zeta)^*(\mathfrak{c}) \subset \mathfrak{a}$. We have proved the following theorem.

Theorem 5.1 *Given are two Lie-Poisson Banach spaces \mathfrak{a} and \mathfrak{c} whose duals are the Banach Lie algebras $\mathfrak{h} := \mathfrak{a}^*$ and $\mathfrak{n} := \mathfrak{c}^*$ respectively, a continuous bilinear skew symmetric map $\omega : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{n}$, and a continuous linear map $\varphi : \mathfrak{h} \rightarrow \text{aut}(\mathfrak{n})$ satisfying (4.5) and (4.6). The Banach space $\mathfrak{c} \oplus \mathfrak{a}$ is an extension of \mathfrak{c} by \mathfrak{a} if and only if*

$$\varphi(\eta)^*(\mathfrak{c}) \subset \mathfrak{c}, \quad (\varphi(\cdot)\zeta)^*(\mathfrak{c}) \subset \mathfrak{a}, \quad \omega(\eta, \cdot)^*(\mathfrak{c}) \subset \mathfrak{a} \quad (5.5)$$

for all $\eta \in \mathfrak{h} = \mathfrak{a}^*$ and $\zeta \in \mathfrak{n} = \mathfrak{c}^*$.

Given $f \in C^\infty(\mathfrak{c} \oplus \mathfrak{a})$, define the partial functional derivatives $\delta f / \delta c \in \mathfrak{c}^*$ and $\delta f / \delta a \in \mathfrak{a}^*$ by

$$D_{\mathfrak{c}} f(c, a)(c') = \left\langle c', \frac{\delta f}{\delta c} \right\rangle \quad \text{and} \quad D_{\mathfrak{a}} f(c, a)(a') = \left\langle a', \frac{\delta f}{\delta a} \right\rangle$$

for all $c' \in \mathfrak{c}$ and all $a' \in \mathfrak{a}$, where $D_{\mathfrak{c}} f(c, a)$ and $D_{\mathfrak{a}} f(c, a)$ denote the partial Fréchet derivatives of f at $(c, a) \in \mathfrak{c} \oplus \mathfrak{a}$ respectively.

Theorem 2.1, (4.4), and (5.4) thus immediately yield the following theorem.

Theorem 5.2 *With the notations and hypotheses of Theorem 5.1, the Lie-Poisson bracket of $f, g \in C^\infty(\mathfrak{c} \oplus \mathfrak{a})$ is given by*

$$\begin{aligned} \{f, g\}(c, a) = & \left\langle a, \left[\frac{\delta f}{\delta a}, \frac{\delta g}{\delta a} \right] \right\rangle \\ & + \left\langle c, \left[\frac{\delta f}{\delta c}, \frac{\delta g}{\delta c} \right] - \varphi \left(\frac{\delta g}{\delta a} \right) \frac{\delta f}{\delta c} + \varphi \left(\frac{\delta f}{\delta a} \right) \frac{\delta g}{\delta c} + \omega \left(\frac{\delta f}{\delta a}, \frac{\delta g}{\delta a} \right) \right\rangle \end{aligned} \quad (5.6)$$

for $c \in \mathfrak{c}$ and $a \in \mathfrak{a}$. The Hamiltonian vector field of $h \in C^\infty(\mathfrak{c} \oplus \mathfrak{a})$ is given by

$$X_h(c, a) = - \left(\text{ad}_{\frac{\delta h}{\delta c}}^* c + \varphi \left(\frac{\delta h}{\delta a} \right)^* c, \omega \left(\frac{\delta h}{\delta a}, \cdot \right)^* c - \left(\varphi(\cdot) \frac{\delta h}{\delta c} \right)^* c + \text{ad}_{\frac{\delta h}{\delta a}}^* a \right) \quad (5.7)$$

Example 1: Semidirect products of Banach Lie-Poisson spaces. Let us apply Theorem 5.1 for the case $\omega = 0$. Condition (4.5) is in this case vacuous and condition (4.6) asserts that $\varphi : \mathfrak{h} \rightarrow \text{aut}(\mathfrak{n})$ is a Lie algebra homomorphism. One can define the **semidirect product of \mathfrak{h} with \mathfrak{n}** as the Banach Lie algebra with underlying Banach space $\mathfrak{n} \oplus \mathfrak{h}$ and bracket (4.4) with $\omega = 0$.

Denote, as before, by \mathfrak{a} and \mathfrak{c} the predual spaces of \mathfrak{h} and \mathfrak{n} respectively, that is, $\mathfrak{h} := \mathfrak{a}^*$ and $\mathfrak{n} := \mathfrak{c}^*$. In this case, only two conditions in Theorem 5.1 survive, namely,

$$\varphi(\eta)^*(\mathfrak{c}) \subset \mathfrak{c} \quad \text{and} \quad (\varphi(\cdot)\zeta)^*(\mathfrak{c}) \subset \mathfrak{a} \quad (5.8)$$

for all $\eta \in \mathfrak{h} = \mathfrak{a}^*$ and $\zeta \in \mathfrak{n} = \mathfrak{c}^*$. The Banach space $\mathfrak{c} \oplus \mathfrak{a}$ is predual to $\mathfrak{n} \oplus \mathfrak{h}$. It is a Banach Lie-Poisson space relative to the Poisson bracket (5.6) and Hamiltonian vector field formula (5.7) with $\omega = 0$. This is the **semidirect product Banach Lie-Poisson space of \mathfrak{a} with \mathfrak{c}** .

An important particular case of this situation occurs when \mathfrak{n} is an Abelian Lie algebra, that is, $\varphi : \mathfrak{h} \rightarrow L^\infty(\mathfrak{n})$ is a Lie algebra representation; $L^\infty(\mathfrak{n})$ denotes the Banach algebra of all bounded linear operators on \mathfrak{n} . In this case one obtains the semidirect product of \mathfrak{h} with the Banach space \mathfrak{n} whose bracket is given by

$$[(\zeta, \eta), (\zeta', \eta')] = (\varphi(\eta)(\zeta') - \varphi(\eta')(\zeta), [\eta, \eta'])$$

for any $\zeta, \zeta' \in \mathfrak{n}$ and $\eta, \eta' \in \mathfrak{h}$. If conditions (5.8) hold, one obtains the semidirect product Banach Lie-Poisson space $\mathfrak{c} \oplus \mathfrak{a}$. The formula for the Poisson bracket is (5.6) with $\omega = 0$ and the first summand of the second term set also equal to zero. The formula for the Hamiltonian vector field is (5.7) with $\omega = 0$ and the first term in the first component set equal to zero. These formulas coincide with the ones found, for example, in Holm, Marsden, and Ratiu [1998].

As a further special case, let us assume that \mathfrak{h} is a W^* -algebra \mathfrak{m} and that \mathfrak{n} is a Hilbert space \mathcal{H} . Additionally, let us fix a W^* -representation $\varphi : \mathfrak{m} \rightarrow L^\infty(\mathcal{H})$ of \mathfrak{m} on the Hilbert space \mathcal{H} . In this case, $\mathfrak{c} = \mathcal{H}$ and $\mathfrak{a} = \mathfrak{m}_*$, where $\mathcal{H} \cong \mathcal{H}_* \cong \mathcal{H}^*$ is equipped with the trivial Poisson structure, since we consider \mathcal{H} as an Abelian Banach Lie algebra. Conditions (5.5) of Theorem 5.1 reduce in this case to the single requirement

$$(\varphi(\cdot)v)^*\mathcal{H} \subset \mathfrak{m}_* \quad \text{for all} \quad v \in \mathcal{H}.$$

This condition can be expressed as follows: for any $v, w \in \mathcal{H}$, there exists an element $b \in \mathfrak{m}_*$ such that

$$\langle \varphi(x)v \mid w \rangle = \langle x, b \rangle,$$

for any $x \in \mathfrak{m}$, where $\langle \cdot \mid \cdot \rangle$ denotes the inner product on \mathcal{H} . But this condition is satisfied since the representation φ is σ -continuous (by definition) and thus the linear functional $x \mapsto \langle \varphi(x)v \mid w \rangle$ is σ -continuous too. This shows that it is represented by an element $b \in \mathfrak{m}_*$ (Sakai [1971]). Therefore we have constructed the semidirect product Banach Lie-Poisson space $\mathcal{H} \oplus \mathfrak{m}_*$.

Let us further specialize this situation for the case $\mathfrak{m} = L^\infty(\mathcal{H})$. The predual space \mathfrak{m}_* is in this case the Banach space of trace class operators $L^1(\mathcal{H})$ and the duality pairing between $L^\infty(\mathcal{H})$ and $L^1(\mathcal{H})$ is given by $\text{tr}(\rho X)$ for $\rho \in L^1(\mathcal{H})$ and $X \in L^\infty(\mathcal{H})$. Formula (5.6) for $f, g \in C^\infty(\mathcal{H} \oplus L^1(\mathcal{H}))$ becomes

$$\{f, g\}(v, \rho) = \text{tr} \left(\rho \left[\frac{\delta f}{\delta \rho}, \frac{\delta g}{\delta \rho} \right] \right) + \left\langle v \mid \frac{\delta f}{\delta \rho} \frac{\delta g}{\delta v} - \frac{\delta g}{\delta \rho} \frac{\delta f}{\delta v} \right\rangle,$$

where $\rho \in L^1(\mathcal{H})$ and $v \in \mathcal{H}$. Hamilton's equation $\dot{f} = \{f, h\}$ for the Hamiltonian $h \in C^\infty(L^1(\mathcal{H}))$ can be equivalently written as the system of equations

$$|\dot{v}\rangle = - \left(\frac{\delta h}{\delta \rho} \right)^* |v\rangle \quad \dot{\rho} = \left[\frac{\delta h}{\delta \rho}, \rho \right] + \left| \frac{\delta h}{\delta \rho} \right\rangle \langle v|.$$

Example 2: An extension of the restricted Banach Lie-Poisson space. Let \mathcal{H} be a complex separable Hilbert space endowed with a *polarization* (Pressley and Segal [1986], Wurzbacher [1990]), that is, a direct sum decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into two closed orthogonal subspaces. Denote by $P_\pm : \mathcal{H} \rightarrow \mathcal{H}_\pm$ the orthogonal projectors on \mathcal{H}_\pm ; hence $P_+ + P_- = \text{id}$ and $P_+P_- = P_-P_+ = 0$. Denote by $L^\infty(\mathcal{H})$ and $L^\infty(\mathcal{H}_\pm)$ the Banach Lie algebra of bounded linear operators on \mathcal{H} and \mathcal{H}_\pm respectively, relative to the commutator bracket. Let $L^2(\mathcal{H})$ be the Banach Lie algebra of linear Hilbert-Schmidt operators on \mathcal{H} , also relative to the commutator bracket. Similarly, let $L^2(\mathcal{H}_+, \mathcal{H}_-)$ and $L^2(\mathcal{H}_-, \mathcal{H}_+)$ be the Banach spaces of Hilbert-Schmidt operators from \mathcal{H}_+ to \mathcal{H}_- and \mathcal{H}_- to \mathcal{H}_+ respectively. Following Pressley and Segal [1986] we call

$$\begin{aligned} \mathfrak{h} := L^\infty(\mathcal{H}, \mathcal{H}_+) := \{ & X \in L^\infty(\mathcal{H}) \mid P_\pm X P_\pm \in L^\infty(\mathcal{H}_\pm), \\ & P_+ X P_- \in L^2(\mathcal{H}_-, \mathcal{H}_+), P_- X P_+ \in L^2(\mathcal{H}_+, \mathcal{H}_-) \} \end{aligned} \quad (5.9)$$

the *restricted Banach Lie algebra*. In this definition we write, for example, $P_+ X P_-$ for $P_+ X P_-|_{\mathcal{H}_-}$ and similarly for the other terms. The vector space \mathfrak{h} is a Banach space relative to the norm

$$\|X\| := \|P_+ X P_+\|_\infty + \|P_- X P_-\|_\infty + \|P_+ X P_-\|_2 + \|P_- X P_+\|_2, \quad (5.10)$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_2$ denote the operator norm and the Hilbert-Schmidt norm in the various spaces. It is easy to show that relative to the commutator bracket $[X, X'] :=$

$XX' - X'X$, the space \mathfrak{h} is a Banach Lie algebra. It is also convenient to think of elements of \mathfrak{h} as block operators of the form

$$\begin{pmatrix} X_+ & X_{+-} \\ X_{-+} & X_- \end{pmatrix}, \quad (5.11)$$

where $X_{\pm} := P_{\pm}XP_{\pm} \in L^{\infty}(\mathcal{H}_{\pm})$, $X_{+-} := P_+XP_- \in L^2(\mathcal{H}_-, \mathcal{H}_+)$, and $X_{-+} := P_-XP_+ \in L^2(\mathcal{H}_+, \mathcal{H}_-)$.

The Banach space $\mathfrak{n} := L^1(\mathcal{H})$ of trace class operators on \mathcal{H}_+ endowed with the trace norm $\|\cdot\|_1$ and the negative of the commutator bracket $[\rho, \rho'] := -\rho\rho' + \rho'\rho$ is also a Banach Lie algebra. Define $\mathfrak{g} := \mathfrak{n} \oplus \mathfrak{h}$ and

$$\varphi : X \in \mathfrak{h} \mapsto [P_+XP_+, \cdot] \in \text{aut}(\mathfrak{n}) \quad (5.12)$$

$$\omega : (X, X') \in \mathfrak{h} \times \mathfrak{h} \mapsto P_+XP_-X'P_+ - P_+X'P_-XP_+ \in \mathfrak{n}. \quad (5.13)$$

The map φ is linear and continuous and the map ω is bilinear and continuous. These maps also satisfy the identities (4.5) and (4.6). Indeed, to verify (4.6), for arbitrary $\sigma \in L^1(\mathcal{H}_+)$ and $X, X' \in L^{\infty}(\mathcal{H}, \mathcal{H}_+)$, taking into account that the bracket operation on \mathfrak{n} is the negative of the commutator bracket, we have

$$\begin{aligned} & \text{ad}_{\omega(X, X')} \sigma + \varphi([X, X'])\sigma - [\varphi(X), \varphi(X')]\sigma \\ &= -[\omega(X, X'), \sigma] + [P_+[X, X']P_+, \sigma] - [[P_+XP_+, \cdot], [P_+X'P_+, \cdot]]\sigma \\ &= -[P_+XP_-X'P_+ - P_+X'P_-XP_+, \sigma] \\ &\quad + [P_+X(P_+ + P_-)X'P_+ - P_+X'(P_+ + P_-)XP_-, \sigma] \\ &\quad - [P_+XP_+, [P_+X'P_+, \sigma]] + [P_+X'P_+, [P_+XP_+, \sigma]] \\ &= [P_+XP_+X'P_+ - P_+X'P_+XP_+, \sigma] + [P_+XP_+, [\sigma, P_+X'P_+]] \\ &\quad + [P_+X'P_+, [P_+XP_+, \sigma]] \\ &= [\sigma, [P_+X'P_+, P_+XP_+]] + [P_+XP_+, [\sigma, P_+X'P_+]] + [P_+X'P_+, [P_+XP_+, \sigma]] = 0 \end{aligned}$$

by the Jacobi identity.

To verify (4.5) we compute separately the first pair of terms for $X, X', X'' \in \mathfrak{h} = L^{\infty}(\mathcal{H}, \mathcal{H}_+)$ to get

$$\begin{aligned} & \omega([X, X'], X'') - \varphi(X)(\omega(X', X'')) = P_+[X, X']P_-X''P_+ - P_+X''P_-[X, X']P_+ \\ &\quad - [P_+XP_+, P_+X'P_-X''P_+ - P_+X''P_-X'P_+] \\ &= P_+X(P_+ + P_-)X'P_-X''P_+ - P_+X'(P_+ + P_-)XP_-X''P_+ \\ &\quad - P_+X''P_-X(P_+ + P_-)X'P_+ + P_+X''P_-X'(P_+ + P_-)XP_+ \\ &\quad - P_+XP_+X'P_-X''P_+ + P_+XP_+X''P_-X'P_+ \\ &\quad + P_+X'P_-X''P_+XP_+ - P_+X''P_-X'P_+XP_+ \\ &= P_+XP_-X'P_-X''P_+ - P_+X'P_+XP_-X''P_+ - P_+X'P_-XP_-X''P_+ \\ &\quad - P_+X''P_-XP_+X'P_+ - P_+X''P_-XP_-X'P_+ + P_+X''P_-X'P_-XP_+ \\ &\quad + P_+XP_+X''P_-X'P_+ + P_+X'P_-X''P_+XP_+ \end{aligned}$$

Rearrange the terms in the following manner:

$$\begin{aligned}
& \omega([X, X'], X'') - \varphi(X)(\omega(X', X'')) = \\
& P_+ X P_- X' P_- X'' P_+ + P_+ X'' P_- X' P_- X P_+ - P_+ X' P_- X P_- X'' P_+ - P_+ X'' P_- X P_- X' P_+ \\
& + P_+ X P_+ X'' P_- X' P_+ - P_+ X' P_+ X P_- X'' P_+ \\
& + P_+ X' P_- X'' P_+ X P_+ - P_+ X'' P_- X P_+ X' P_+ \\
& = P_+ (X P_- X' P_- X'' + X'' P_- X' P_- X - X' P_- X P_- X'' - X'' P_- X P_- X') P_+ \\
& + P_+ (X P_+ X'' P_- X' - X' P_+ X P_- X'') P_+ \\
& + P_+ (X' P_- X'' P_+ X - X'' P_- X P_+ X') P_+.
\end{aligned}$$

Adding the other two terms obtained by circular permutations gives zero; the summands cancel separately in the three groups emphasized above. Thus Proposition 4.1 can be applied thereby showing that \mathfrak{g} is an extension of $\mathfrak{h} = L^\infty(\mathcal{H}, \mathcal{H}_+)$ by $\mathfrak{n} = L^1(\mathcal{H}_+)$. The Lie bracket on $\mathfrak{n} \oplus \mathfrak{h}$ is given by (4.4) which in this case becomes

$$\begin{aligned}
[(X, \rho), (X', \rho')] &= (-[\rho, \rho'] + \varphi(X)(\rho') - \varphi(X')(\rho) + \omega(X, X'), [X, X']) \\
&= (-[\rho, \rho'] + [P_+ X P_+, \rho'] - [P_+ X' P_+, \rho] \\
&\quad + P_+ X P_- X' P_+ - P_+ X' P_- X P_+, [X, X'])
\end{aligned} \tag{5.14}$$

The predual of \mathfrak{h} is the Banach Lie-Poisson space

$$\begin{aligned}
\mathfrak{a} := L^1(\mathcal{H}, \mathcal{H}_+) &:= \{\sigma \in L^\infty(\mathcal{H}) \mid P_\pm \sigma P_\pm \in L^1(\mathcal{H}_\pm), \\
&\quad P_+ \sigma P_- \in L^2(\mathcal{H}_-, \mathcal{H}_+), P_- \sigma P_+ \in L^2(\mathcal{H}_+, \mathcal{H}_-)\}
\end{aligned} \tag{5.15}$$

relative to the pairing

$$\langle \sigma, X \rangle := \text{trace}(\sigma_+ X_+ + \sigma_- X_- + \sigma_{+-} X_{-+} + \sigma_{-+} X_{+-}) \tag{5.16}$$

for $X \in \mathfrak{h}$. The predual of \mathfrak{n} is $\mathfrak{c} := \mathcal{K}(\mathcal{H}_+)$, the Banach Lie-Poisson space of compact operators on \mathcal{H}_+ . We shall verify now the hypotheses of Theorem 5.1, that is,

- (i) $\varphi(X)^*(\mathfrak{c}) \subset \mathfrak{c}$, where $\varphi(X) : L^1(\mathcal{H}_+) \rightarrow L^1(\mathcal{H}_+)$, so $\varphi(X)^* : L^\infty(\mathcal{H}_+) \rightarrow L^\infty(\mathcal{H}_+)$ and one needs to show $\varphi(X)^*(\mathcal{K}(\mathcal{H}_+)) \subset \mathcal{K}(\mathcal{H}_+)$,
- (ii) $(\varphi(\cdot)\rho)^*(\mathfrak{c}) \subset \mathfrak{a}$, where $\varphi(\cdot)\rho : L^\infty(\mathcal{H}, \mathcal{H}_+) \rightarrow L^1(\mathcal{H}_+)$, so $(\varphi(\cdot)\rho)^* : L^\infty(\mathcal{H}_+) \rightarrow (L^\infty(\mathcal{H}, \mathcal{H}_+))^*$ and one needs to show $(\varphi(\cdot)\rho)^*(\mathcal{K}(\mathcal{H}_+)) \subset L^1(\mathcal{H}, \mathcal{H}_+)$
- (iii) $\omega(X, \cdot)^*(\mathfrak{c}) \subset \mathfrak{a}$, where $\omega(X, \cdot) : L^\infty(\mathcal{H}, \mathcal{H}_+) \rightarrow L^1(\mathcal{H}_+)$, so $\omega(X, \cdot)^* : L^\infty(\mathcal{H}_+) \rightarrow (L^\infty(\mathcal{H}, \mathcal{H}_+))^*$ and one needs to show $\omega(X, \cdot)^*(\mathcal{K}(\mathcal{H}_+)) \subset L^1(\mathcal{H}, \mathcal{H}_+)$

for all $X \in \mathfrak{h} = L^\infty(\mathcal{H}, \mathcal{H}_+)$ and $\rho \in \mathfrak{n} = L^1(\mathcal{H}_+)$.

To verify (i) use the trace pairing, let $X \in \mathfrak{h}$, $Y \in L^\infty(\mathcal{H}_+)$, and $\rho \in L^1(\mathcal{H}_+)$ to get

$$\langle \varphi(X)^* Y, \rho \rangle = \langle Y, \varphi(X) \rho \rangle = \text{trace}(Y[P_+ X P_+, \rho]) = \text{trace}([Y, P_+ X P_+] \rho)$$

which shows that $\varphi(X)^* Y = [Y, P_+ X P_+] \in L^\infty(\mathcal{H}_+)$, that is, $\varphi(X)^* = -[P_+ X P_+, \cdot]$. Therefore, if $\varkappa \in \mathcal{K}(\mathcal{H}_+)$, we have $\varphi(X)^* \varkappa = -[P_+ X P_+, \varkappa] \in \mathcal{K}(\mathcal{H}_+)$, since $\mathcal{K}(\mathcal{H}_+)$ is an ideal in $L^\infty(\mathcal{H}_+)$. This shows that $\varphi(X)^*(\mathfrak{c}) \subset \mathfrak{c}$.

To show (ii), use the same notations as before to get

$$\langle (\varphi(\cdot)\rho)^*Y, X \rangle = \langle Y, \varphi(X)\rho \rangle = \text{trace}(Y[P_+XP_+, \rho]) = \text{trace}([\rho, Y]P_+XP_+).$$

Thus, if $\varkappa \in \mathcal{K}(\mathcal{H}_+)$ we have

$$\langle (\varphi(\cdot)\rho)^*\varkappa, X \rangle = \text{trace}([\rho, \varkappa]P_+XP_+)$$

so, taking for X operators of the form X_- , X_{+-} , and X_{-+} , the right hand side of this relation vanishes. Therefore

$$(\varphi(\cdot)\rho)^*\varkappa = \begin{pmatrix} [\rho, \varkappa] & 0 \\ 0 & 0 \end{pmatrix} \in L^1(\mathcal{H}, \mathcal{H}_+)$$

since $[\rho, \varkappa] \in L^1(\mathcal{H}_+)$.

Finally, to verify (iii), use the same notations as above, let $X' \in L^\infty(\mathcal{H}, \mathcal{H}_+)$ to get

$$\begin{aligned} \langle \omega(X, \cdot)^*Y, X' \rangle &= \langle Y, \omega(X, X') \rangle = \text{trace}(Y(P_+XP_-X'P_+ - P_+X'P_-XP_+)) \\ &= \text{trace}(YX_{+-}X'_{-+} - X_{-+}YX'_{+-}). \end{aligned}$$

Thus if $\varkappa \in \mathcal{K}(\mathcal{H}_+)$, we have

$$\langle \omega(X, \cdot)^*\varkappa, X' \rangle = \text{trace}(\varkappa X_{+-}X'_{-+} - X_{-+}\varkappa X'_{+-})$$

which shows that

$$\omega(X, \cdot)^*\varkappa = \begin{pmatrix} 0 & \varkappa X_{+-} \\ -X_{-+}\varkappa & 0 \end{pmatrix} \in L^1(\mathcal{H}, \mathcal{H}_+)$$

because $\varkappa X_{+-} \in L^2(\mathcal{H}_-, \mathcal{H}_+)$, $X_{-+}\varkappa \in L^2(\mathcal{H}_+, \mathcal{H}_-)$ since $X_{+-} \in L^2(\mathcal{H}_-, \mathcal{H}_+)$, $X_{-+} \in L^2(\mathcal{H}_+, \mathcal{H}_-)$ and the L^2 operators are an ideal in the algebra of bounded operators.

In view of Theorems 5.1 and 5.2 the direct sum $\mathfrak{c} \oplus \mathfrak{a} = \mathcal{K}(\mathcal{H}_+) \oplus L^1(\mathcal{H}, \mathcal{H}_+)$ is a Banach Lie-Poisson space relative to the bracket

$$\begin{aligned} \{f, g\}(\varkappa, \sigma) &= \text{trace} \left(\sigma \left[\frac{\delta f}{\delta \sigma}, \frac{\delta g}{\delta \sigma} \right] \right) \\ &+ \text{trace} \left(\varkappa \left(\left[\frac{\delta f}{\delta \varkappa}, \frac{\delta g}{\delta \varkappa} \right] - \left[P_+ \frac{\delta g}{\delta \sigma} P_+, \frac{\delta f}{\delta \varkappa} \right] + \left[P_+ \frac{\delta f}{\delta \sigma} P_+, \frac{\delta g}{\delta \varkappa} \right] \right. \right. \\ &\quad \left. \left. + P_+ \frac{\delta f}{\delta \sigma} P_- \frac{\delta g}{\delta \sigma} P_+ - P_+ \frac{\delta g}{\delta \sigma} P_- \frac{\delta f}{\delta \sigma} P_+ \right) \right) \end{aligned} \quad (5.17)$$

for $f, g \in C^\infty(\mathcal{K}(\mathcal{H}_+) \oplus L^1(\mathcal{H}, \mathcal{H}_+))$ and $\varkappa \in \mathcal{K}(\mathcal{H}_+)$, $\sigma \in L^1(\mathcal{H}, \mathcal{H}_+)$. The Hamiltonian vector field of $h \in C^\infty(\mathcal{K}(\mathcal{H}_+) \oplus L^1(\mathcal{H}, \mathcal{H}_+))$ is given by

$$\begin{aligned} X_h(\varkappa, \sigma) &= - \left(\left[\varkappa, \frac{\delta h}{\delta \varkappa} \right] + \left[\varkappa, P_+ \frac{\delta h}{\delta \sigma} P_+ \right], \right. \\ &\quad \left(\begin{pmatrix} 0 & \varkappa(\delta h / \delta \sigma)_{+-} \\ -(\delta h / \delta \sigma)_{-+}\varkappa & 0 \end{pmatrix} - \begin{pmatrix} [\delta h / \delta \varkappa, \varkappa] & 0 \\ 0 & 0 \end{pmatrix} + \left[\sigma, \frac{\delta h}{\delta \sigma} \right] \right) \end{aligned} \quad (5.18)$$

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